B.Sc I.

Topic - Theory of Equation

(i) Relations Between the Roots and Co-efficients of Equations.

Let $P_0x^n + P_1x^{n-1} + P_2x^{n-2} + \cdots + P_{n-1}x + P_n = 0$ — (1)

be a polynomial equation of degree n. Let $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_{n-1}, \alpha_n$ be the n roots of the equation

then we have identity

$$P_0x^n + P_1x^{n-1} + P_2x^{n-2} + \cdots + P_{n-1}x + P_n = P_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_{n-1})(x-\alpha_n)$$

$$\Rightarrow x^n + \frac{P_1}{P_0}x^{n-1} + \frac{P_2}{P_0}x^{n-2} + \cdots + \frac{P_{n-1}}{P_0}x + \frac{P_n}{P_0} \equiv (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1})(x-\alpha_n)$$

$$\equiv x^n - \Sigma\alpha\, x^{n-1} + \Sigma\alpha_1\alpha_2\, x^{n-2} - \Sigma\alpha_1\alpha_2\alpha_3\, x^{n-3} + \cdots + (-1)^n\alpha_1\alpha_2\alpha_3\cdots\alpha_n$$

$$= x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \cdots + (-1)^n S_n.$$

Where S_r is the sum of all products of $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ taken r at a time

On equating the co-efficients of terms containing like powers of x on the two sides of the above identity

$S_1 = \Sigma\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ = Sum of the roots taken one at a time

$= -\frac{P_1}{P_0} = (-1)^1\frac{P_1}{P_0}$

$S_2 = \Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \cdots + \alpha_2\alpha_n = \frac{P_2}{P_0} = (-1)^2\frac{P_2}{P_0}$

= Sum of the product of roots taken two at a time.

$S_3 = \sum \alpha_1 \alpha_2 \alpha_3 = (-1)^3 \frac{P_3}{P_0}$

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$S_r = \sum \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_r = (-1)^r \frac{P_r}{P_0}$

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$S_n = \alpha_1 \alpha_2 \cdots \alpha_n = (-1)^n \frac{P_n}{P_0}$

Cor :→ 1. If $P_0 x^2 + P_1 x + P_2 = 0$ be a quadratic equation and say α_1, α_2 be its roots

then $\sum \alpha_1 = \alpha_1 + \alpha_2 = (-1)^1 \frac{P_1}{P_0}$

$\alpha_1 \alpha_2 = (-1)^2 \frac{P_2}{P_0}$

2. If $P_0 x^3 + P_1 x^2 + P_2 x + P_3 = 0$ be the cubic equation. If $\alpha_1, \alpha_2, \alpha_3$ are its roots

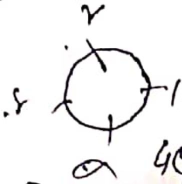
then $\alpha_1 + \alpha_2 + \alpha_3 = (-1)^1 \frac{P_1}{P_0} = -\frac{P_1}{P_0}$

$\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_1 \alpha_3 = (-1)^2 \frac{P_2}{P_0} = \frac{P_2}{P_0}$

$\alpha_1 \alpha_2 \alpha_3 = (-1)^3 \frac{P_3}{P_0} = -\frac{P_3}{P_0}$

3. If $P_0 x^4 + P_1 x^3 + P_2 x^2 + P_3 x + P_4 = 0$ be a Biquadratic equation and $\alpha, \beta, \gamma, \delta$ are its roots

$^4C_1 = 4$



$^4C_2 = \frac{4!}{2!\,2!} = 6$

$^4C_3 = \frac{4!}{3!\,1!}$

then $\alpha + \beta + \gamma + \delta = (-1) \frac{P_1}{P_0} = -\frac{P_1}{P_0}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \sum \alpha\beta = (-1)^2 \frac{P_2}{P_0} = \frac{P_2}{P_0}$

$\sum \alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = (-1)^3 \frac{P_3}{P_0} = -\frac{P_3}{P_0}$

$\alpha\beta\gamma\delta = (-1)^4 \frac{P_4}{P_0} = -\frac{P_4}{P_0}$

Problem :→

Solve the Equation $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$

Whose roots are in G.P

Here Given Equation $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$

is Biquadratic Equation and therefore it has four roots

Let $\alpha, \beta, \gamma, \delta$ are its roots

and these are in G.P.

Say $\alpha = \frac{a}{r^3}$, $\beta = \frac{a}{r}$, $\gamma = ar$, $\delta = ar^3$

Now $\alpha\beta\gamma\delta = (-1)^4\, 64 = 64$

$\Rightarrow \frac{a}{r^3} \times \frac{a}{r} \cdot ar \cdot ar^3 = 64$

$\Rightarrow a^4 = 64 = 8^2$

$\Rightarrow (a^2)^2 = 8^2$

$\Rightarrow a^2 = 8$ ———— (1)

Also $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = (-1)^2 \cdot 70 = 70.$

$\frac{a}{r^3} \cdot \frac{a}{r} + \frac{a}{r^3} \cdot ar + \frac{a}{r^3} \cdot ar^3 + \frac{a}{r} \cdot ar + \frac{a}{r} \cdot ar^3 + ar \cdot ar^3 = 70$

$\Rightarrow \frac{a^2}{r^4} + \frac{a^2}{r^2} + a^2 + a^2 + a^2r^2 + a^2r^4 = 70$

$\Rightarrow a^2\left[\frac{1}{r^4} + \frac{1}{r^2} + 1 + 1 + r^2 + r^4\right] = 70$

$\Rightarrow 8\left[\frac{1}{r^4} + r^4 + r^2 + \frac{1}{r^2} + 2\right] = 70$

$\frac{1}{r^4} + r^4 + 2 + r^2 + \frac{1}{r^2} = \frac{70}{8}$

$\Rightarrow \left(r^2 + \frac{1}{r^2}\right)^2 + \left(r^2 + \frac{1}{r^2}\right) = \frac{70}{8}$

put $r^2 + \frac{1}{r^2} = x$

$\Rightarrow x^2 + x = \frac{70}{8}$

$\Rightarrow x^2 + x - \frac{70}{8} = 0$ this is a quadratic Equation in x

Ex:→ Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P and hence solve the equation

① $x^3 - 12x^2 + 39x - 28 = 0$

Solⁿ

Here Given Equation $x^3 - px^2 + qx - r = 0$ is a cubic Equation.

Since by the Question its roots are in A.P.

Therefore Say $\alpha-\beta, \alpha, \alpha+\beta$ be the roots of the equation.

Now Sum of the roots =

$\alpha-\beta+\alpha+\alpha+\beta = (-1)^1(-p) = p$

$\Rightarrow 3\alpha = p \Rightarrow \alpha = \frac{p}{3}$

As α is a root of the equation.

this $\alpha^3 - p\alpha^2 + q\alpha - r = 0$

$\left(\frac{p}{3}\right)^3 - p\cdot\left(\frac{p}{3}\right)^2 + q\cdot\frac{p}{3} - r = 0$

$\Rightarrow \frac{p^3}{27} - \frac{p^3}{9} + \frac{pq}{3} - r = 0$

$\Rightarrow p^3 - 3p^3 + 9pq - 27r = 0$

$\Rightarrow -2p^3 + 9pq - 27r = 0$

$\Rightarrow 2p^3 - 9pq + 27r = 0$

is the required condition.

Next. Consider the equation $x^3 - 12x^2 + 39x - 28 = 0$

Let $\alpha-\beta, \alpha, \alpha+\beta$ are its roots

$\therefore 3\alpha = 12 \Rightarrow \alpha = 4$

Next part-

product of the roots

$(\alpha-\beta)\times\alpha\cdot(\alpha+\beta) = -(-28)$

$\Rightarrow (\alpha^2-\beta^2) = \frac{28}{\alpha} = \frac{28}{4} = 7$

$16 - \beta^2 = 7$

$\beta^2 = 9$

$\beta = \pm 3.$

$\alpha-\beta = 4 \mp 3 = 1, 7$

$\alpha = 4$

$\alpha+\beta = 4 \pm 3 = 7, 1$

$\therefore$ Roots of the equation are 1, 4, 7

Now we have.

$\alpha+\beta+\gamma = (-1)^1 \cdot \frac{1}{1} = -1$

$\alpha\beta+\beta\gamma+\gamma\alpha = (-1)^2\, 2 = 2$

$\alpha\beta\gamma = (-1)^3\, 8 = -8$

By the question roots are in G.P

Say $\alpha = \frac{a}{r}$ $\beta = a$ $\gamma = ar$

$\therefore \alpha\beta\gamma = -8$

$\Rightarrow \frac{a}{r}\cdot a \cdot ar = -8$

$\Rightarrow a^3 = -8 = (-2)^3$

$\Rightarrow a = -2$

Als. $\alpha+\beta+\gamma = -1$

$\Rightarrow \frac{a}{r} + a + ar = -1$

$a\left(\frac{1}{r}+1+r\right) = -1$

$r + \frac{1}{r} + 1 = -\frac{1}{a} = \frac{1}{2}$

$\Rightarrow \frac{r^2+1+r}{r} = \frac{1}{2}$

$\Rightarrow 2r^2+2r+2 = r$

$\Rightarrow 2r^2+r+2 = 0$ As it is a quadratic Equation in r

$\Rightarrow r = \frac{-1 \pm \sqrt{1-16}}{4}$

$= \frac{-1\pm\sqrt{-15}}{4} = \frac{-1\pm\sqrt{15}\, i}{4}$

$\therefore$ Required roots $\alpha = \frac{a}{r} = \frac{-2}{\frac{-1-\sqrt{15}\,i}{4}}$.

$\beta = a = -2$

$\gamma = ar = -2 \times \frac{-1+i\sqrt{15}}{4}$

$\therefore$ i.e the roots are..

$\alpha = \frac{a}{r} = \frac{-2\times 4(-1+\sqrt{15}\,i)}{1-15i^2} = \frac{-8(-1+\sqrt{15}\,i)}{16}$ ...

$\beta = a = -2$

$\gamma = ar = \frac{1-i\sqrt{15}}{2}$